JOURNAL OF APPROXIMATION THEORY 60, 157–167 (1990)

On Even-Degree Splines with Application to Quadratures

M. N. EL TARAZI AND A. A. KARABALLI

Mathematics Department, Kuwait University, P. O. Box 5969, Safat, 13060 Kuwait

Communicated by R. Bojanic

Received February 1, 1988

A class of splines of even degree $k = 2\alpha$ and continuity order $\mathscr{C}^\alpha$ that match the derivatives up to order α at the knots of a uniform partition are studied for $\alpha = 1, 2, 3, 4$, and 5. The simple direct formulas obtained can be applied to quadratures.

1. INTRODUCTION

Recently, El Tarazi and Sallam [3] have constructed an interpolatory quartic spline which matches the first and second derivatives of a given function at the knots.

In this paper we extend that work, studying a class of splines of even degree $k = 2\alpha$ and continuity order $\mathscr{C}^\alpha$ that match the derivatives up to the order α at the knots of a uniform partition for $\alpha = 1, 2, 3, 4$, and 5. The reason for restricting ourselves to even-degree splines is that the formulas obtained are explicit. There are no linear systems to solve.

In Section 2 we study the construction, existence, uniqueness, and error bounds for the proposed splines. In Section 3 some conjectures relating these different splines are stated. Finally, in Section 4 we apply these splines to quadratures. Both theory and numerical results show the method to be efficient.

2. SPLINES OF DEGREE 2, 4, 6, 8, AND 10

We construct here a class of interpolating splines of degree k, for $k = 2, 4, 6, 8$, and 10. $\mathscr{L}_\infty$ error estimates for these splines are also represented. Since all cases considered are similar, details are given only for the case of degree $k = 6$.

0021-9045/90 $3.00

Let $\{x_i, i=0, 1, ..., N+1\}$ be a uniform partition of $[0, 1]$. Set $h=x_{i+1}-x_i$ for $i=0, 1, ..., N$. ($g_i^{(r)}$ stands for $g^{(r)}(x_i)$, $i=0, 1, ..., N+1$ and $r=0, 1, ...$.) We have the following cases:

2.1. *Spline of Degree 2*

Given the real numbers f_i' $(i=0, 1, ..., N+1)$ and f_0, there exists a unique spline $s(x) \in \mathscr{C}^1[0, 1]$ of degree 2 (a polynomial of degree 2 in each subinterval $[x_i, x_{i+1}]$) such that

$$\begin{aligned} s_i' &= f_i' \qquad (i=0, 1, ..., N+1) \\ s_0 &= f_0. \end{aligned} \tag{1}$$

For a fixed $i \in \{0, 1, ..., N\}$, set $x=x_i+th$, $0 \leqslant t \leqslant 1$. In $[x_i, x_{i+1}]$ the spline $s(x)$ of degree 2 satisfying (1) is

$$s(x)=s_i A_0(t)+s_{i+1} A_1(t)+hf_i' A_2(t) \tag{2}$$

with

$$A_0(t)=-t^2+1, \qquad A_1(t)=t^2, \qquad A_2(t)=-t^2+t, \tag{3}$$

where s_i $(i=1, 2, ..., N+1)$ are easily computed throughout the recurrence formula

$$2(-s_{i-1}+s_i)=h(f_{i-1}'+f_i'), \qquad s_0=f_0. \tag{4}$$

In this case we have, for any $x \in [0, 1]$, the error bounds

$$\begin{aligned} |s^{(r-1)}(x)-f^{(r+1)}(x)| &\leqslant \frac{h^{2-r}}{(4)^{1-r} r!\,(2-2r)!} \|f^{(3)}\|_\infty, \qquad r=0, 1 \\ |s(x)-f(x)| &\leqslant \frac{h^2}{4 \cdot 2!} \|f^{(3)}\|_\infty \end{aligned} \tag{5}$$

provided $f \in \mathscr{C}^3[0, 1]$. (Details are given for the similar case $k=6$.)

2.2. *Spline of Degree* 4

This particular case is included, with slightly different assumptions (f_{N+1} is given instead of f_{N+1}''), in the work of El Tarazi and Sallam [3]. Given the real numbers f_i', f_i'' $(i=0, 1, ..., N+1)$, and f_0, there exists a unique spline $s(x) \in \mathscr{C}^2[0, 1]$ of degree 4 (a polynomial of degree 4 in each subinterval $[x_i, x_{i+1}]$) such that

$$\begin{aligned} &s_i' = f_i', \qquad s_i'' = f_i'' \qquad (i = 0, 1, ..., N+1) \\ &s_0 = f_0. \end{aligned} \tag{6}$$

For a fixed $i \in \{0, 1, ..., N\}$, set $x = x_i + th$, $0 \leqslant t \leqslant 1$. In $[x_i, x_{i+1}]$ the spline $s(x)$ of degree 4 satisfying (6) is

$$s(x) = s_i A_0(t) + s_{i+1} A_1(t) + h[f_i' A_2(t) + f_{i+1}' A_3(t)] + h^2 f_i'' A_4(t) \tag{7}$$

with

$$\begin{aligned} A_0(t) &= 3t^4 - 4t^3 + 1 \\ A_1(t) &= -3t^4 + 4t^3 \\ A_2(t) &= 2t^4 - 3t^3 + t \\ A_3(t) &= t^4 - t^3 \\ A_4(t) &= (t^4 - 2t^3 + t^2)/2, \end{aligned} \tag{8}$$

where s_i $(i = 1, 2, ..., N+1)$ are easily computed throughout the recurrence formula

$$12(-s_{i-1} + s_i) = 6h(f_{i-1}' + f_i') + h^2(f_{i-1}'' - f_i''), \qquad s_0 = f_0. \tag{9}$$

We have in this case, for any $x \in [0, 1]$, the error bounds

$$\begin{aligned} |s^{(r+1)}(x) - f^{(r+1)}(x)| &\leqslant \frac{h^{4-r}}{(4)^{2-r} r!\,(4-2r)!} \|f^{(5)}\|_\infty, \qquad r = 0, 1, 2 \\ |s(x) - f(x)| &\leqslant \frac{h^4}{4^2 \cdot 4!} \|f^{(5)}\|_\infty \end{aligned} \tag{10}$$

provided $f \in \mathscr{C}^5[0, 1]$. (Details are given for the similar case $k = 6$.)

2.3. *Spline of Degree* 6

Given the real numbers $f_i', f_i'', f_i^{(3)}$ $(i = 0, 1, ..., N+1)$, and f_0, there exists a unique spline $s(x) \in \mathscr{C}^3[0, 1]$ of degree 6 (a polynomial of degree 6 in each subinterval $[x_i, x_{i+1}]$) such that

$$\begin{aligned} &s_i' = f_i', \qquad s_i'' = f_i'', \qquad s_i^{(3)} = f_i^{(3)} \qquad (i = 0, 1, ..., N+1) \\ &s_0 = f_0. \end{aligned} \tag{11}$$

Indeed we can express any polynomial $p(t)$ in $[0, 1]$ of degree 6 in terms of its values at 0 and 1, its first and second derivatives at 0 and 1, and its third derivative at 0,

$$p(t) = p_0 A_0(t) + p_1 A_1(t) + p_0' A_2(t) + p_1' A_3(t) + p_0'' A_4(t) + p_1'' A_5(t) + p_0^{(3)} A_6(t).$$

To determine $A_0, A_1, ..., A_6$, we write the above equality for $p(t) = 1, t, t^2, ..., t^6$. We get the linear system

$$\begin{aligned}
&A_0 + A_1 &&= 1\\
&A_1 + A_2 + A_3 &&= t\\
&A_1 + 2A_3 + 2A_4 + 2A_5 &&= t^2\\
&A_1 + 3A_3 + 6A_5 + 6A_6 &&= t^3\\
&A_1 + 4A_3 + 12A_5 &&= t^4\\
&A_1 + 5A_3 + 20A_5 &&= t^5\\
&A_1 + 6A_3 + 30A_5 &&= t^6.
\end{aligned}$$

Solving this simple system we get

$$\begin{aligned}
A_0(t) &= -10t^6 + 24t^5 - 15t^4 + 1\\
A_1(t) &= 10t^6 - 24t^5 + 15t^4\\
A_2(t) &= -6t^6 + 15t^5 - 10t^4 + t\\
A_3(t) &= -4t^6 + 9t^5 - 5t^4\\
A_4(t) &= (-3t^6 + 8t^5 - 6t^4 + t^2)/2\\
A_5(t) &= (t^6 - 2t^5 + t^4)/2\\
A_6(t) &= (-t^6 + 3t^5 - 3t^4 + t^3)/6.
\end{aligned} \tag{12}$$

Now for a fixed $i \in \{0, 1, ..., N\}$, set $x = x_i + th$, $0 \leqslant t \leqslant 1$. In $[x_i, x_{i+1}]$ the spline $s(x)$ of degree 6 satisfying (11) is

$$s(x) = s_i A_0(t) + s_{i+1} A_1(t) + h[f_i' A_2(t) + f_{i+1}' A_3(t)] + h^2[f_i'' A_4(t) + f_{i+1}'' A_5(t)] + h^3 f_i^{(3)} A_6(t). \tag{13}$$

We have a similar expression for $s(x)$ in $[x_{i-1}, x_i]$. Since $s(x) \in \mathscr{C}^3[0, 1]$

so $s^{(3)}(x_i^-)=s^{(3)}(x_i^+)$ $(i=1, 2, ..., N)$ and $s^{(3)}(x_{N+1}^-)=f_{N+1}^{(3)}$ lead to the recurrence formula

$$120(-s_{i-1}+s_i)=60h(f'_{i-1}+f'_i)+12h^2(f''_{i-1}-f''_i)$$
$$+h^3(f_{i-1}^{(3)}+f_i^{(3)}), \qquad s_0=f_0. \tag{14}$$

In order to give an error bound for the above spline and its derivatives, we recall first the following result due to Ciarlet *et al.* [2].

Let $g\in\mathscr{C}^{2m}[0, h]$ be given. Let p_{2m-1} be the unique Hermite interpolation polynomial of degree $2m-1$ that matches g and its first $m-1$ derivatives $g^{(r)}$ at 0 and h. Then

$$|e^{(r)}(x)|\leqslant\frac{h^r[x(h-x)]^{m-r}G}{r!\,(2m-2r)!}, \qquad r=0, 1, ..., m; \qquad 0\leqslant x\leqslant h, \tag{15.1}$$

where

$$|e^{(r)}(x)|=|g^{(r)}(x)-p_{2m-1}^{(r)}(x)| \qquad \text{and} \qquad G=\max_{0\leqslant x\leqslant h}|g^{(2m)}(x)|. \tag{15.2}$$

The bounds in (15.1) are best possible for $r=0$ only. For some values of m ($m=2$ and $m=3$) optimal error bounds on the derivatives $e^{(r)}(x)$ do exist (see Birkhoff and Priver [1], or Varma and Howell [4]).

Now we go back to our spline. Notice that is $[x_i, x_{i+1}]$ $(i=0, 1, ..., N)$, $s'(x)$ is the Hermite interpolation polynomial of degree 5 matching f', f'', $f^{(3)}$ at x_i and x_{i+1}, so for any $x\in[x_i, x_{i+1}]$ we have [using (15.1) with $m=3$, $g=f'$, and $p_5=s'$]

$$|s^{(r-1)}(x)-f^{(r+1)}(x)|\leqslant\frac{h^r[(x-x_i)(x_{i+1}-x)]^{3-r}}{r!\,(6-2r)!}\|f^{(7)}\|_\infty, \qquad r=0, 1, 2, 3.$$

It follows then that

$$|s^{(r-1)}(x)-f^{(r+1)}(x)|\leqslant\frac{h^r[h^2q(1-q)]^{3-r}}{r!\,(6-2r)!}\|f^{(7)}\|_\infty, \qquad r=0, 1, 2, 3$$

with $q=(x-x_i)/h$. Therefore for any $x\in[0, 1]$ we have

$$|s^{(r+1)}(x)-f^{(r+1)}(x)|\leqslant\frac{h^{6-r}}{(4)^{3-r}r!\,(6-2r)!}\|f^{(7)}\|_\infty, \qquad r=0, 1, 2, 3 \tag{15.3}$$

provided $f\in\mathscr{C}^7[0, 1]$. Integrating over $[0, x]$ (for $r=0$), using $s(0)=f(0)$, we obtain

$$|s(x)-f(x)|\leqslant\frac{h^6}{4^3\cdot 6!}\|f^{(7)}\|_\infty. \tag{15.4}$$

2.4. *Spline of Degree* 8

Given the real numbers f_i', f_i'', $f_i^{(3)}$, $f_i^{(4)}$ $(i=0, 1, ..., N+1)$, and f_0, there exists a unique spline $s(x) \in \mathscr{C}^4[0, 1]$ of degree 8 (a polynomial of degree 8 in each subinterval $[x_i, x_{i+1}]$) such that

$$s_i' = f_i', \qquad s_i'' = f_i'', \qquad s_i^{(3)} = f_i^{(3)}, \qquad s_i^{(4)} = f_i^{(4)} \qquad (i=0, 1, ..., N+1)$$
$$s_0 = f_0. \tag{16}$$

For a fixed $i \in \{0, 1, ..., N\}$, set $x = x_i + th$, $0 \leqslant t \leqslant 1$. In $[x_i, x_{i+1}]$, the spline $s(x)$ of degree 8 satisfying (16) is

$$\begin{aligned} s(x) = {} & s_i A_0(t) + s_{i+1} A_1(t) + h[f_i' A_2(t) + f_{i+1}' A_3(t)] \\ & + h^2[f_i'' A_4(t) + f_{i+1}'' A_5(t)] \\ & + h^3[f_i^{(3)} A_6(t) + f_{i+1}^{(3)} A_7(t)] + h^4 f_i^{(4)} A_8(t) \end{aligned} \tag{17}$$

with

$$\begin{aligned} A_0(t) &= 35t^8 - 120t^7 + 140t^6 - 56t^5 + 1 \\ A_1(t) &= -35t^8 + 120t^7 - 140t^6 + 56t^5 \\ A_2(t) &= 20t^8 - 70t^7 + 84t^6 - 35t^5 + t \\ A_3(t) &= 15t^8 - 50t^7 + 56t^6 - 21t^5 \\ A_4(t) &= (10t^8 - 36t^7 + 45t^6 - 20t^5 + t^2)/2 \\ A_5(t) &= (-5t^8 + 16t^7 - 17t^6 + 6t^5)/2 \\ A_6(t) &= (4t^8 - 15t^7 + 20t^6 - 10t^5 + t^3)/6 \\ A_7(t) &= (t^8 - 3t^7 + 3t^6 - t^5)/6 \\ A_8(t) &= (t^8 - 4t^7 + 6t^6 - 4t^5 + t^4)/24, \end{aligned} \tag{18}$$

where s_i $(i=1, 2, ..., N+1)$ are easily computed using the recurrence formula

$$\begin{aligned} 1680(-s_{i-1} + s_i) = {} & 840h(f_{i-1}' + f_i') + 180h^2(f_{i-1}'' - f_i'') \\ & + 20h^3(f_{i-1}^{(3)} + f_i^{(3)}) + h^4(f_{i-1}^{(4)} - f_i^{(4)}), \qquad s_0 = f_0. \end{aligned} \tag{19}$$

We have in this case, for any $x \in [0, 1]$, the error bounds

$$\begin{aligned} |s^{(r+1)}(x) - f^{(r+1)}(x)| &\leqslant \frac{h^{8-r}}{(4)^{4-r} r!\,(8-2r)!} \|f^{(9)}\|_\infty, \qquad r = 0, 1, 2, 3, 4 \\ |s(x) - f(x)| &\leqslant \frac{h^8}{4^4 \cdot 8!} \|f^{(9)}\|_\infty \end{aligned} \tag{20}$$

provided $f \in \mathscr{C}^9[0, 1]$. (Details are given for the similar case $k = 6$.)

2.5. *Spline of Degree* 10

Given the real numbers f_i', f_i'', $f_i^{(3)}$, $f_i^{(4)}$, $f_i^{(5)}$ $(i=0, 1, ..., N+1)$, and f_0, there exists a unique spline $s(x) \in \mathscr{C}^5[0, 1]$ of degree 10 (a polynomial of degree 10 in each subinterval $[x_i, x_{i-1}]$) such that

$$s_i' = f_i',\ s_i'' = f_i'',\ s_i^{(3)} = f_i^{(3)},\ s_i^{(4)} = f_i^{(4)},\ s_i^{(5)} = f_i^{(5)}\ \ (i=0, 1, ..., N+1)$$
$$s_0 = f_0. \tag{21}$$

For a fixed $i \in \{0, 1, ..., N\}$, set $x = x_i + th$, $0 \leqslant t \leqslant 1$. In $[x_i, x_{i+1}]$ the spline $s(x)$ of degree 10 satisfying (21) is

$$\begin{aligned} s(x) = {} & s_i A_0(t) + s_{i+1} A_1(t) + h[f_i' A_2(t) + f_{i+1}' A_3(t)] \\ & + h^2[f_i'' A_4(t) + f_{i+1}'' A_5(t)] \\ & + h^3[f_i^{(3)} A_6(t) + f_{i+1}^{(3)} A_7(t)] \\ & + h^4[f_i^{(4)} A_8(t) + f_{i+1}^{(4)} A_9(t)] + h^5 f_i^{(5)} A_{10}(t) \end{aligned} \tag{22}$$

with

$$\begin{aligned} A_0(t) &= -126t^{10} + 560t^9 - 945t^8 + 720t^7 - 210t^6 + 1 \\ A_1(t) &= 126t^{10} - 560t^9 + 945t^8 - 720t^7 + 210t^6 \\ A_2(t) &= -70t^{10} + 315t^9 - 540t^8 + 420t^7 - 126t^6 + t \\ A_3(t) &= -56t^{10} + 245t^9 - 405t^8 + 300t^7 - 84t^6 \\ A_4(t) &= (-35t^{10} + 160t^9 - 280t^8 + 224t^7 - 70t^6 + t^2)/2 \\ A_5(t) &= (21t^{10} - 90t^9 + 145t^8 - 104t^7 + 28t^6)/2 \\ A_6(t) &= (-15t^{10} + 70t^9 - 126t^8 + 105t^7 - 35t^6 + t^3)/6 \\ A_7(t) &= (-6t^{10} + 25t^9 - 39t^8 + 27t^7 - 7t^6)/6 \\ A_8(t) &= (-5t^{10} + 24t^9 - 45t^8 + 40t^7 - 15t^6 + t^4)/24 \\ A_9(t) &= (t^{10} - 4t^9 + 6t^8 - 4t^7 + t^6)/24 \\ A_{10}(t) &= (-t^{10} + 5t^9 - 10t^8 + 10t^7 - 5t^6 + t^5)/120, \end{aligned} \tag{23}$$

where s_i $(i=1, 2, ..., N+1)$ are easily computed throughout the recurrence formula

$$\begin{aligned} 30240(-s_{i-1} + s_i) = {} & 15120h(f_{i-1}' + f_i') \\ & + 3360h^2(f_{i-1}'' - f_i'') + 420h^3(f_{i-1}^{(3)} + f_i^{(3)}) \\ & + 30h^4(f_{i-1}^{(4)} - f_i^{(4)}) + h^5(f_{i-1}^{(5)} + f_i^{(5)}), \quad s_0 = f_0. \end{aligned} \tag{24}$$

We have in this case, for any $x \in [0, 1]$, the error bounds

$$|s^{(r+1)}(x) - f^{(r+1)}(x)| \leqslant \frac{h^{10-r}}{(4)^{5-r} r!\,(10-2r)!} \|f^{(11)}\|_\infty, \qquad r = 0, 1, 2, 3, 4, 5$$
$$|s(x) - f(x)| \leqslant \frac{h^{10}}{4^5 \cdot 10!} \|f^{(11)}\|_\infty \tag{25}$$

provided $f \in \mathscr{C}^{11}[0, 1]$. (Details are given for the similar case $k = 6$.)

3. CONJECTURES

Consider Table I obtained from (3), (8), and (12) after factorization. It is easily seen that the polynomials A_0, A_2, A_4, and A_6 follow a specific simple pattern according to which, for case $k = 8$, we should expect A_0, A_2, A_4, A_6, and A_8 to be

$$A_0(t) = (t-1)^4(35t^4 + 20t^3 + 10t^2 + 4t + 1)$$
$$A_2(t) = \frac{t}{1!}(t-1)^4(20t^3 + 10t^2 + 4t + 1)$$
$$A_4(t) = \frac{t^2}{2!}(t-1)^4(10t^2 + 4t + 1)$$
$$A_6(t) = \frac{t^3}{3!}(t-1)^4(4t + 1)$$
$$A_8(t) = \frac{t^4}{4!}(t-1)^4,$$

TABLE I

k	A_0	A_2	A_4	A_6
2	$-(t-1)(t+1)$	$-\frac{t}{1!}(t-1)$		
4	$(t-1)^2(3t^2+2t+1)$	$\frac{t}{1!}(t-1)^2(2t+1)$	$\frac{t^2}{2!}(t-1)^2$	
6	$-(t-1)^3(10t^3+6t^2+3t+1)$	$-\frac{t}{1!}(t-1)^3(6t^2+3t+1)$	$-\frac{t^2}{2!}(t-1)^3(3t+1)$	$-\frac{t^3}{3!}(t-1)^3$

which is correct. The factor $(35t^4+20t^3+10t^2+4t+1)$ is deduced from $(10t^3+6t^2+3t+1)$ as follows: $4=1+3$, $10=1+3+6$, $20=1+3+6+10$, and $35=1+4+10+20$. Now from (3), (8), (12), and (18), after factorization, we have Table II. A_1, A_3, A_5, and A_7 follow a specific pattern but it is more difficult to see. In fact one should first compute $A_1=1-A_0$ and then start to deduce the other polynomials. For the case $k=8$ we find that

$$120=(15\times 8)/1, \qquad 140=(35\times 8)/2, \qquad 56=(21\times 8)/3$$
$$35=(\ 5\times 7)/1, \qquad 21=(\ 6\times 7)/2.$$

To see this more clearly consider the case $k=10$. From (23), after factorization, we have

$$A_1(t)=\frac{t^6}{0!}(t-1)^0(126t^4-560t^3+945t^2-720t+210)=1-A_0(t)$$

$$A_3(t)=-\frac{t^6}{1!}(t-1)^1(56t^3-189t^2+216t-84)$$

$$A_5(t)=\frac{t^6}{2!}(t-1)^2(21t^2-48t+28)$$

$$A_7(t)=-\frac{t^6}{3!}(t-1)^3(6t-7)$$

$$A_9(t)=\frac{t^6}{4!}(t-1)^4.$$

TABLE II

k	A_1	A_3	A_5	A_7
2	t^2			
4	$-t^3(3t-4)$	$\frac{t^3}{1!}(t-1)$		
6	$t^4(10t^2-24t+15)$	$-\frac{t^4}{1!}(t-1)(4t-5)$	$\frac{t^4}{2!}(t-1)^2$	
8	$-t^5(35t^3-120t^2+140t-56)$	$\frac{t^5}{1!}(t-1)(15t^2-35t+21)$	$-\frac{t^5}{2!}(t-1)^2(5t-6)$	$\frac{t^5}{3!}(t-1)^3$

Now let us note the following:

$$560 = (56 \times 10)/1, \qquad 945 = (189 \times 10)/2,$$
$$189 = (21 \times 9)/1, \qquad 216 = (48 \times 9)/2,$$
$$48 = (6 \times 8)/1, \qquad 28 = (7 \times 8)/2,$$
$$720 = (216 \times 10)/3, \qquad 210 = (84 \times 10)/4.$$
$$84 = (28 \times 9)/3,$$

From (4), (9), (14), (19), and (24) we see that the coefficients (from left to right) involved in these recurrence formulas follow the pattern

k						
2	2!/1!	1!/(0! 1!)				
4	4!/2!	3!/(1! 1!)	2!/(0! 2!)			
6	6!/3!	5!/(2! 1!)	4!/(1! 2!)	3!/(0! 3!)		
8	8!/4!	7!/(3! 1!)	6!/(2! 2!)	5!/(1! 3!)	4! (0! 4!)	
10	10!/5!	9!/(4! 1!)	8!/(3! 2!)	7!/(2! 3!)	6!(1! 4!)	5!/(0! 5!)

Finally, it is clear that (5), (10), (15), (20), and (25) follow a simple pattern.

4. Application

As an interesting application, the above splines constitute a new class of numerical quadrature rules since they allow us to approximate

$$f(x) = \int_a^x f'(t)\, dt \qquad \text{in } [a, b], \tag{26}$$

an integral which appears often in statistics when computing distributions.

Notice that (4), applied to the function f given in (26), is the classical trapezoidal rule, while (9), (14), (19), and (24) are the classical trapezoidal rules with end correction.

We applied all of the above formula on

$$f(x) = \int_1^x dt/(t+1) \qquad \text{in } [1, 5].$$

TABLE III

N	k:2	4	6	8	10
1	6.9E-02	6.1E-03	6.2E-04	6.8E-05	7.6E-06
5	8.2E-03	9.7E-05	1.5E-06	2.5E-08	4.5E-10
10	2.5E-03	8.9E-06	4.2E-08	2.3E-10	1.3E-12
15	1.2E-03	2.0E-06	4.5E-09	1.2E-11	3.2E-14
20	6.8E-04	6.8E-07	8.8E-10	1.4E-12	2.7E-15
25	4.4E-04	2.9E-07	2.5E-10	2.5E-13	6.0E-16
30	3.1E-04	1.5E-07	8.6E-11	6.0E-14	5.7E-16

For each case we first computed s_i $(i = 1, 2, ..., N+1)$ by the corresponding recurrence formula. Then we computed the value of $s(x)$ at N equally spaced points in each subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, ..., N)$ by its corresponding expression. [For instance, for $k = 8$, (19) is first used, then (17).] This was done on a personal computer using a simple Basic program in double precision. When dealing with polynomials, nested form may reduce the effect of round-off errors. [$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ in nested form is $(((a_4x + a_3)x + a_2)x + a_1)x + a_0$.]

Table III of bounds for the actual error shows the method to be effective and confirms the theoretical results.

5. Conclusion

We have studied the existence and uniqueness of a class of splines of even degree that match the derivatives at the knots to a given order, obtaining direct simple formulas. Error estimates were derived which, together with the numerical results, showed the method to be efficient.

References

1. G. Birkhoff and A. Priver, Hermite interpolation errors for derivatives, *J. Math. Phys.* **46** (1967), 440–447.
2. P. G. Ciarlet, M. H. Schultz, and R. S. Varga, Numerical methods of high-order accuracy, *Numer. Math.* **9** (1967), 394–430.
3. M. N. El Tarazi and S. Sallam, On quartic splines with application to quadratures. *Computing* **38** (1987), 355–361.
4. A. K. Varma and G. Howell, Best error bounds for derivatives in two point Birkhoff interpolation problems, *J. Approx. Theory* **38** (1983), 258–268.